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A FINITE-DIMENSIONAL NORMED SPACE WITH TWO NON-EQUIVALENT SYMMETRIC BASES

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ABSTRACT

A sequence of finite-dimensional normed spaces is constructed, each with two symmetric bases, such that the sequence of equivalence constants between these bases is unbounded. An essential tool in the proof is the edge-isoperimetric inequality in the discrete cube.

In 1981, Read [9] disproved a long-standing conjecture that any Banach space either has at most one symmetric basis, up to equivalence, or else has uncountably many. He showed that a Banach space can have any finite number of non-equivalent symmetric bases, or can have a countably infinite collection of them. The natural finite-dimensional analogue of this question, asked by Johnson, Maurey, Schechtman and Tzafriri [7], is the following. Does there exist a function $f\colon [1,\infty) \to [1,\infty)$ such that any two C-symmetric bases of the same finite-dimensional normed space are always $f(C)$-equivalent? The most general result known in this direction is due to Schütt [10], who showed that, for every $\alpha > 0$ and every $C > 1$, there exists $\gamma = \gamma(\alpha, C)$ such that if X is any n-dimensional normed space with $d(X, \ell_2^n) \geq n^\alpha$, then any two symmetric bases $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ of X are necessarily $\gamma(\alpha, C)$-equivalent. Other positive results concerning the uniqueness of symmetric and unconditional bases have been proved by Bourgain, Casazza, Lindenstrauss and Tzafriri [3], by Casazza, Kalton and Tzafriri [4] and in the paper of Johnson, Maurey, Schechtman and Tzafriri mentioned earlier. In this paper we give an example of a sequence of finite-dimensional spaces with two symmetric bases that are not uniformly equivalent, answering

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in the negative the question above. The construction is fairly simple: it is likely that a more complicated variant of it would give a better estimate for how great the lack of equivalence can be. We shall discuss this matter briefly at the end of the paper.

Let us begin by describing our space in general terms. Let A be an orthogonal matrix (the work will consist later in finding an orthogonal matrix with suitable properties) and let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by A. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ and let Ω be the group of symmetries of ℓ_1^n, that is, the group of linear maps of the form

$$\sum_{i=1}^{n} a_i \mathbf{e}_i \mapsto \sum_{i=1}^{n} \epsilon_i a_i \mathbf{e}_{\pi(i)}$$

where $\epsilon_i = \pm 1$ for each i and $\pi \in S_n$. For $1 \le i \le n$ let $\mathbf{e}_i' = \alpha \mathbf{e}_i$. This will be our second basis. Finally let Ψ be the obvious corresponding group of symmetries associated with this basis, that is, $\Psi = \{\alpha\omega\alpha^{-1}\colon \omega \in \Omega\}$.

Given a map α, the norm is constructed as follows. Define X_0 to be the set $\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\}$. Then define sets $X_1, X_2, \ldots$ inductively as follows. If j is odd, then

$$X_j = \{\psi x : x \in X_{j-1}, \psi \in \Psi\},$$

and if j is even, then

$$X_j = \{\omega x\colon x \in X_{j-1}, \omega \in \Omega\}.$$

Given $x \in \mathbb{R}^n$, we now define $\|x\|$ to be $\max\{2^{-j}|\langle x, x_j\rangle|\colon x_j \in X_j, j \in \mathbb{N}\}$. Obviously, this maximum is really over a finite set, but this is not important.

It is easy to show that the bases $(\mathbf{e}_i)_{i=1}^n$ and $(\mathbf{e}_i')_{i=1}^n$ are both 2-symmetric. Indeed, suppose that $\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}_i$, that $\|\mathbf{a}\| = 2^{-j}|\langle \mathbf{a}, x\rangle|$ for some $x \in X_j$ and that j is even. By the definition of X_j, $x = \omega y$ for some $y \in X_{j-1}$ and $\omega \in \Omega$, from which it follows that $\omega' x = (\omega'\omega^{-1})\omega y$ is in X_j for every $\omega' \in \Omega$. This implies that

$$\|\omega' a\| \ge 2^{-j}|\langle \omega'\mathbf{a}, \omega' x\rangle| = 2^{-j}|\langle \mathbf{a}, x\rangle| = \|\mathbf{a}\|$$

for every $\omega' \in \Omega$.

If on the other hand j is odd, then for every $\omega \in \Omega$, we have $\omega x \in X_{j+1}$, so that

$$\|\omega\mathbf{a}\| \ge 2^{-(j+1)}|\langle \omega\mathbf{a}, \omega x\rangle| = 2^{-(j+1)}|\langle \mathbf{a}, x\rangle| = 2^{-1}\|\mathbf{a}\|.$$

It follows that $(\mathbf{e}_i)_{i=1}^n$ is 2-symmetric. The proof for $(\mathbf{e}'_i)_{i=1}^n$ is almost identical.

The idea now is to choose α so that $\|\mathbf{e}'_1\| \equiv \|\alpha\mathbf{e}_1\|$ is as small as possible. If it tends to zero, then we are done, because it is easy to show that the two bases are at best $(1/\|\mathbf{e}'_1\|)$-equivalent, by comparing $\|\mathbf{e}_1\|^{-1}\mathbb{E}(\sum_{i=1}^n g_i\mathbf{e}_i)$ with $\|\mathbf{e}'_1\|^{-1}\mathbb{E}(\sum_{i=1}^n g_i\mathbf{e}'_i)$, where $g_1,\ldots,g_n$ are i.i.d. Gaussian random variables. It is also not hard to show that the construction just outlined must work for some (possibly not orthogonal) linear map α if there is any example at all, but we will not go into this.

It turns out that the ideal properties for the matrix A of α to have are the following. First, it should have rather few non-zero entries in each row and column, and second, it should be fairly random in appearance (so, for example, a tensor product of a $k\times k$-Walsh matrix with the identity on $\mathbb{R}^{n/k}$ would be no good). We shall begin by exhibiting a symmetric $2^k\times 2^k$ matrix which has k entries of $\pm k^{-1/2}$ in each row (and column) with the property that the supports of any two rows are either disjoint or intersect in exactly two places. Obviously, they cannot intersect in one place only and have an inner product of zero.

This matrix can be constructed in various different ways. The first is inductive. Define $A'_0 = (1)$ and, for $k>0$, let A'_k be defined by

$$A'_k = \begin{pmatrix} A'_{k-1} & I_{k-1} \\ I_{k-1} & -A'_{k-1}\end{pmatrix}$$

where I_{k-1} is the $2^{k-1}\times 2^{k-1}$-identity matrix. Then, for $k>0$, let $A_k = k^{-1/2}A'_k$, and let α_k be the linear map on $\mathbb{R}^{2^k}$ defined by A_k. It is easy to check that A_k is an orthogonal matrix; since it is symmetric, we also have $\alpha_k^2 = 1$.

The second construction is a geometrical one—as the adjacency matrix of a graph with signed edges. The graph is just the k-dimensional cube, that is, the power set of $[k]\equiv\{1,2,\ldots,k\}$, with $A,B\subset[k]$ joined by an edge if and only if $|A\,\triangle\,B| = 1$. The signs on the edges are defined inductively as follows. Let Q_1 have a positive sign on its single edge. Now, having determined signs for all the edges of Q_k, divide the vertices of Q_{k+1} into two classes, namely

$$\mathcal{A}_0 = \{A\subset[k+1] : k+1\notin A\}$$

and

$$\mathcal{A}_1 = \{A\subset[k+1] : k+1\in A\}.$$

Then the subgraphs of Q_{k+1} induced by $\mathcal{A}_0$ and $\mathcal{A}_1$ are each naturally isomorphic to Q_k. Let $\mathcal{A}_0$ be given the signs of Q_k (according to the natural isomorphism) and let $\mathcal{A}_1$ be given the opposite signs. Finally, let the edges between $\mathcal{A}_0$ and $\mathcal{A}_1$ all have positive signs. Again, it is not hard to verify that, after a suitable normalization, the adjacency matrix of this signed graph is orthogonal and symmetric, either directly, or by observing that it is the same matrix as A_k defined earlier.

Our task now is to estimate $\|\mathbf{e}_1'\|$, where $\mathbf{e}_1'$ and $\|.\|$ depend in the way outlined earlier on the map $\alpha = \alpha_k$ just defined. Of key importance is the following lemma, due to Harper [5], Bernstein [1], Hart [6] and Lindsey [8]. (See [2] for a particularly short proof.)

LEMMA 1: *Let $\mathcal{A}, \mathcal{B} \subset Q_k$ be two subsets of the vertices of the k-cube of cardinality r. Then the number of edges joining a vertex in $\mathcal{A}$ to a vertex in $\mathcal{B}$ is at most $r \log_2 r$.*

Note that if $r = 2^l$ for some integer l, then this maximum is attained by setting $\mathcal{A} = \mathcal{B} = \{A \subset [k] \colon A \subset [l]\}$.

The next lemma is the main technical lemma after which our proof will be easy. First, let us set $n = 2^k$ and define a sequence of subsets $Y_1, Y_2, \ldots$ of the unit sphere of ℓ_2^n as follows. If j is even, then $Y_j = X_j$, where X_j is defined as earlier in terms of α. If j is odd, then $Y_j = \{\omega\alpha x : \omega \in \Omega, x \in Y_{j-1}\}$.

LEMMA 2: *For any $i \in \mathbb{N}$ with $k^i \le n$ let $y_i = k^{-1/2} \sum_{t=1}^{k^i} \mathbf{e}_t$. Then if $k^j \le n$ and $x \in Y_j$, we have*

$$|\langle x, \omega y_i\rangle| \le f(j) k^{-1/2}$$

for every $\omega \in \Omega$ and every y_i for which $|i-j|$ is odd, where $f(j) = j!(2\log_2 k)^{2j}$.

Proof: We shall use induction on j. When $j = 0$, the result is trivial. Let us suppose then that j is odd, that the result is true for $j-1$ and that $x \in Y_{j-1}$. Note that

$$\max\{|\langle \omega\alpha x, y_i\rangle| \colon \omega \in \Omega\} = \max\{|\langle \alpha x, \omega y_i\rangle| \colon \omega \in \Omega\}$$

for every x and every i. We would therefore like to show that $|\langle \alpha x, \omega y_i\rangle| \le f(j)k^{-1/2}$ for every even value of i and every $\omega \in \Omega$.

Without loss of generality we may suppose that the coordinates $(x_i)_{i=1}^n$ of x (with respect to $(\mathbf{e}_i)_{i=1}^n$) are non-negative and in non-increasing order. It follows from the inductive hypothesis and this assumption that $\sum_{t=1}^{k^i} x_t \le k^{i/2} f(j) k^{-1/2}$

for every odd value of i. We therefore have also that $x_t \le f(j)t^{-1}k^{-(i+1)/2}$, for every pair t, i of integers such that i is even and $t \le k^i \le n$. That is, we have that $x_t \le f(j)\lambda(t)$ for every $1 \le t \le n$, where

$$\lambda(t) = \begin{cases} t^{-1}k^{i/2} & i = \lfloor(\log t/\log k)\rfloor \text{ even,} \\ k^{-(i+1)/2} & i = \lfloor(\log t/\log k)\rfloor \text{ odd.} \end{cases}$$

We shall now estimate $|\langle \alpha x, \omega y_i\rangle|$ for an even value of i. First, let us write $x = x^{(1)} + x^{(2)} + x^{(3)}$, where

$$x^{(1)} = \sum_{t=1}^{k^{i-1}} x_t \mathbf{e}_t,$$
$$x^{(2)} = \sum_{t=k^{i-1}+1}^{k^{i+1}} x_t \mathbf{e}_t,$$
$$x^{(3)} = \sum_{t=k^{i+1}+1}^{n} x_t \mathbf{e}_t.$$

Then, by the Cauchy–Schwarz inequality, the fact that $\|x\|_2 = 1$ and the estimate for $\lambda(t)$, we have

$$\begin{aligned} \left\|x^{(1)}\right\|_1 &\le \left\|\sum_{t=1}^{k^{i-2}} x_t \mathbf{e}_t\right\|_1 + \left\|\sum_{t=k^{i-2}+1}^{k^{i-1}} x_t \mathbf{e}_t\right\|_1 \\ &\le k^{(i-2)/2} + f(j-1)k^{(i-2)/2}\log k \\ &\le 2f(j-1)k^{(i-2)/2}\log k \end{aligned}$$

and hence, since $\|\alpha\|_1 = \sqrt{k}$, that $\left\|\alpha x^{(1)}\right\|_1 \le 2f(j-1)k^{(i-2)/2}\log k$. It follows that $|\langle \alpha x^{(1)}, \omega y_i\rangle| \le 2f(j-1)k^{-1/2}\log k$.

The estimate for $|\langle \alpha x^{(3)}, \omega y_i\rangle|$ is also easy. Indeed, $\left\|x^{(3)}\right\|_\infty \le k^{-(i+2)/2}f(j-1)$ and $\|\alpha\|_\infty = \sqrt{k}$, so $\left\|\alpha x^{(3)}\right\|_\infty \le k^{-(i+1)/2}f(j-1)$, which implies that

$$\begin{aligned} |\langle \alpha x^{(3)}, \omega y_i\rangle| &\le k^{-i/2}k^{-(i+1)/2}k^i f(j-1) \\ &\le k^{-1/2}f(j-1)\ . \end{aligned}$$

Of course, the most important estimate is that of $|\langle \alpha x^{(2)}, \omega y_i\rangle|$, and this is where we shall use Lemma 1. Note that the support of x has cardinality at most

k^j, so if $i > j$ then $x^{(2)} = 0$. Otherwise, observe that $x^{(2)}$ is dominated pointwise by $f(j-1)\sum_{s=0}^{k}\xi^{(s)}$, where

$$\xi^{(s)} = \begin{cases} k^{-i/2}\sum_{t=1}^{k^i} \mathbf{e}_t, & s = 1, \\ (s-1)^{-1}k^{-i/2}\sum_{t=(s-1)k^i+1}^{sk^i} \mathbf{e}_t, & 2 \le s \le k. \end{cases}$$

Now, if A and B are two subsets of $[n]$ corresponding to subsets $\mathcal{A}$ and $\mathcal{B}$ of the vertices of Q_k, and if χ_A and χ_B are the characteristic functions of A and B, but with arbitrary signs on the coordinates, then $|\langle\alpha\chi_A, \chi_B\rangle|$ is certainly bounded above by the number of edges between $\mathcal{A}$ and $\mathcal{B}$ in Q_k, multiplied by $k^{-1/2}$. Hence, using Lemma 1, we obtain

$$|\langle\alpha\xi^{(s)}, \omega y_i\rangle| \le \begin{cases} k^{-i/2}k^{-i/2}k^{-1/2}\log_2(k^i)k^i, & s = 1, \\ k^{-i/2}(s-1)^{-1}k^{-i/2}k^{-1/2}\log_2(k^i)k^i, & 2 \le s \le k. \end{cases}$$

Adding, we obtain

$$\begin{aligned} |\langle\alpha x^{(2)}, \omega y_i\rangle| &\le k^{-1/2} i \log_2 k\Big(1 + \sum_{s=1}^{k-1} s^{-1}\Big) f(j-1) \\ &\le 2f(j-1)k^{-1/2}j(\log_2 k)^2 \ , \end{aligned}$$

since, as remarked earlier, we may as well assume that $i \le j$.

Putting these three estimates together, we obtain that

$$\begin{aligned} |\langle\alpha x^{(2)}, \omega y_i\rangle| &\le 3k^{-1/2}f(j-1)j(\log_2 k)^2 \\ &= f(j)k^{-1/2} \end{aligned}$$

as required.

This completes the inductive step when j is odd. When j is even, the argument is almost identical, so we shall not give it. ∎

We are now ready to prove our main result.

THEOREM 3: *Let $n = 2^k$ for some integer k. There exists an n-dimensional normed space with two 2-symmetric bases $(\mathbf{e}_i)_{i=1}^n$ and $(\mathbf{e}'_i)_{i=1}^n$ whose constant of equivalence is at least* $\exp(\log\log n / 8\log\log\log n)$.

Proof: As remarked earlier, it is enough to show, if we take the bases $(\mathbf{e}_i)_{i=1}^n$, $(\mathbf{e}'_i)_{i=1}^n$ and the norm defined in terms of $\alpha = \alpha_k$ above, that the norm of the vector $\|\mathbf{e}'_1\|$ is at most $\le \exp(-\log\log n/8\log\log\log n)$.

Let $X_0, X_1, X_2, \ldots$ and $Y_0, Y_1, Y_2, \ldots$ be as defined earlier, and note that $X_j = Y_j$ if j is even and $X_j \subset Y_{j+1}$ if j is odd. Hence, by Lemma 2,

$$\begin{aligned}\|\mathbf{e}_1'\| &= \max\{2^{-j}|\langle \mathbf{e}_1', x_j\rangle| : j \in \mathbb{N}, x_j \in X_j\}\\ &\leq \max_{j\in\mathbb{N}} 2^{-j}\Big[f(j+1)k^{-1/2} \wedge 1\Big].\end{aligned}$$

By considering the ratio of $f(j+1)$ to $f(j)$ it is easy to see that the second maximum is achieved either when j is as large as possible such that $f(j+1) < k^{1/2}$ or when it is as small as possible such that $f(j+1) > k^{1/2}$. In either case we have $(j+2)!(\log_2 k)^{2(j+2)} \geq k^{1/2}$ from which it is easy to deduce that $j \geq \log_2 k/5\log_2\log_2 k$. It follows that

$$\begin{aligned}\|\mathbf{e}_1'\| &\leq 2^{-\log_2 k/5\log_2\log_2 k}\\ &\leq \exp(-\log\log n/8\log\log\log n)\end{aligned}$$

as was needed. ∎

It is not hard to get an estimate for the distance of our space from ℓ_2^n. We sketch the argument. Since α is orthogonal, $\|\alpha\|_{\ell_1^n\to\ell_1^n} = k$ and $\|\omega\|_{\ell_2^n\to\ell_2^n} = \|\omega\|_{\ell_1^n\to\ell_1^n} = 1$ for every $\omega \in \Omega$, it follows that if $x_j \in X_j$, then $\|x_j\|_2 = 1$ and $\|x_j\|_1 \leq k^{j+1}$. Hence, for any $x \in \mathbb{R}^n$,

$$\|x\| \leq \max_{j\geq 0} 2^{-j}\left[k^{j+1}\|x\|_\infty \wedge 1\right].$$

Again, it is easy to see, to within 1, for which value of j the maximum occurs. It will certainly satisfy $k^{-1} \leq k^{j+1}\|x\|_\infty$, that is, $k^{j+2} \geq \|x\|_\infty^{-1}$. If $x = (\pm n^{-1/2}, \ldots, \pm n^{-1/2})$, then we get $n^{1/2} \leq k^{j+2}$, from which it follows easily that $\|x\| \leq \exp\Big(-\log n/8\log\log n\Big)$, and hence that the distance of our space from ℓ_2^n is at least $\exp\Big(\log n/8\log\log n\Big)$. It is straightforward to obtain an upper bound of roughly this size as well (that is, with some other constant replacing 8).

If one is interested in maximizing the equivalence constant between two symmetric bases (i.e., in making them as non-equivalent as possible) then the main weakness in the construction of this paper is that we have no control over the degree of the vertices of the graph whose adjacency matrix we used. It is not hard to see that a much larger degree would give a better result, at least if it was possible to obtain a lemma along the lines of Lemma 1. If k is the degree and n the number of vertices, then the method could in principle give a bound of about $\max\{k^{1/2}, \exp(\log n/\log k)\}$ which is largest when $k = \exp(\sqrt{\log n})$ or so. The distance of the space from ℓ_2^n would be of the order of $\exp(\log n/\log k)$, so it would be smaller, the larger the lack of equivalence, as one would expect from Schütt's result.

However, all we have managed to prove so far is that a very natural class of graphs cannot provide substantially better examples. A regular graph G with the property that any two distinct elements $x, y \in G$ have either zero or two neighbours in common is known to design theorists as a **semibiplane**. A wide variety of semibiplanes can be constructed as follows. Regard Q_k as a k-dimensional vector space over the field with two elements in the obvious way, and let $x_1, \dots, x_r$ be r elements of the space. An r-regular graph can be constructed by joining $x \in Q_k$ to all vectors of the form $x + x_i$ for $1 \leq i \leq r$. (This is just the Cayley graph with generators $x_1, \dots, x_r$). It is easy to see that the graph is connected iff $x_1, \dots, x_r$ is a linearly independent sequence. It is not hard to see also that for many choices of $x_1, \dots, x_r$, suitably separated in some sense, the resulting graph is a semibiplane. Let us call the graph obtained from Q_k and $x_1, \dots, x_r$ in this way $G(k; x_1, \dots, x_r)$. Then the following result holds.

PROPOSITION 4: *Suppose that the edges of $G(k; x_1, \dots, x_r)$ can be given signs in such a way that the (signed) adjacency matrix is orthogonal. Then $r \leq 2k$.*

In other words, the best we can do with graphs of the form $G(k; x_1, \dots, x_r)$ is to double the number of non-zero entries in each row and column of the adjacency matrix. This provides very weak evidence that Theorem 3 may be best possible. If this is the case, then it seems that a deep proof will be necessary. On the other hand, there are various constructions of semibiplanes not based on the cube (the standard reference is [11]), and even the rather special combinatorial problem of whether their edges can be given signs so that their signed adjacency matrices become orthogonal is an interesting one.

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